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Supersymmetric Integrable Systems in (2 + 1) Dimensions and Their Backlund Transformation

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Nonlinear integrable systems in (2 + 1) dimensions which are supersymmetric are generated in two different ways. In one approach the homogeneous spaces of super-Lie algebra are used, and in the other we use a different technique of extending the dimension of the system. The two sets of equations turn out to be different. The methodologies of Darbux–Backlund transformation and gauge transformation are used to generate the Backlund transformations of these equations. An important result of our analysis is the existence of purely fermionic nonlinear systems in (2 + 1) dimensions.

1. INTRODUCTION

Nonlinear integrable systems in one space and one time dimension have been exhaustively studied. An important aspect of present research is to extend the class of integrable hierarchies to (2 + 1) dimensions. Important contributions in this direction have been made by Ablowitz,(1) Fokas,(2) Zakharov,(3) and others.(4) The supersymmetric generalizations of nonlinear systems in (1 + 1) dimensions and their various properties are being studied, but supersymmetric systems in (2 + 1) dimensions have not been dealt with in such detail. In this paper we construct nonlinear integrable systems in (2 + 1) dimensions which are supersymmetric in two different ways. It is observed that these two sets of nonlinear systems are different. In one approach we use the spectral parameter-dependent Lax pair in (2 + 1) dimensions utilizing the idea of Fordy and Kulish(6) in conjunction with the homogeneous space idea of super-Lie algebra.(7) In the other approach we use the technique of Zakharov.(8) Solutions for the nonlinear equations so obtained can be studied

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with the help of Backlund transformations[9] derived by two different methods for the two different Lax operators.

2. FORMULATION

To start with we adopt the method of homogeneous spaces of Lie algebra which was initially used by Fordy and Kulish. A homogeneous space of a Lie group G is any differentiable manifold M on which G acts transitively. If K is an isotropy group of G, then M can be identified with a coset G/K. Let us denote by g and k the corresponding Lie algebras. Then,

$$g = k \oplus m$$

and

$$[k, k] \subset k; \qquad [k, m] \subset m; \qquad [m, m] \subset k$$

For our present discussion we consider a super-Lie algebra g whose Z_2 grading implies

$$\begin{aligned} g &= g^{\text{even}} + g^{\text{odd}} \\ g^{\text{even}} &= \{H_i, E_A, E_D\} \\ g^{\text{odd}} &= \{F_i, F_j\} \end{aligned} \tag{1}$$

where H_i are the commuting generators, E_A, E_D are step operators, and F_i, F_j are their fermionic counterparts. So we consider the required Lax operators in the following form (A belonging to the Cartan subalgebra):

$$\begin{aligned} \phi_x &= [\lambda \cdot A + Q]\phi \\ \phi_t &= \lambda^{n-1}\, \phi_y + \sum_{i=0}^{n-2} B_{n-i}(x, y, t)\lambda^i \phi \end{aligned} \tag{2}$$

where A, Q, B_{n-i} are matrices belonging to the super-Lie algebra. Integrability conditions imposed on (2) (we henceforth consider for simplicity $n = 2$) yield

$$\begin{aligned} \partial_y Q &= [A, B_2] \\ \partial_x B_{n-i} &= [A, B_{n-i+1}] + [Q, B_{n-i}] \\ \partial_t Q &= \partial_x B_n - [Q, B_n] \end{aligned} \tag{3}$$

where all the commutators written above are to be interpreted in their graded form: $[A, B] = AB - (-1)^{p(A)p(B)}\, BA$, where $p(A)$, $p(B)$ are the parities of the corresponding matrices or generators. Let us now assume that

$$g = k \oplus m$$

and hence $B_i = B_i^k + B_i^m$ with the stipulation that $B_i^k \in k$, $B_i^m \in m$. Then equation (3) immediately leads to

$$\partial_y Q = [A, B_2^m] \tag{4a}$$

$$\partial_x B_{n-i}^k = [Q, B_{n-i}^m] \tag{4b}$$

$$\partial_t Q = \partial_x B_n^m - [Q, B_n^k] \tag{4c}$$

along with

$$\partial_x B_{n-i}^m = [A, B_{n-i+1}^m] + [Q, B_{n-i}^k], \qquad i = 1, \ldots, n-2 \tag{4d}$$

To proceed further let us make a choice of the super-Lie algebra g and the subspaces k and m. The Lie algebra $OSP(2/1)$ is generated by five generators: (H, E, F), which are even, and (P, R), which are odd. The basic commutation and anticommutation rules are

$$\begin{aligned} &[H, E]_- = 2E, \qquad [H, P]_- = P, \qquad [E, R]_- = P \\ &[R, R]_+ = 2F, \qquad [H, F]_- = -2F, \qquad [H, R]_- = -R, \qquad [F, P]_- = R \\ &[P, R]_+ = H, \qquad [E, F]_- = H, \qquad [P, P]_+ = -2E \end{aligned} \tag{5}$$

These suggest that

$$k = \{H, E, F\}, \qquad m = \{P, R\} \tag{6}$$

with the property that $[k, k] \subset k$, $[k, m] \subset m$, and $[m, m] \subset k$. So we set

$$Q = q_1 P + q_2 R \tag{7}$$

and substituting in equation (4) for $n = 2$, we get

$$\begin{aligned} q_{1t} &= q_{1xy} + q_1 \partial_x^{-1}(q_2 q_{1y} - q_1 q_{2y}) - 2 q_2 \partial_x^{-1}(q_1 q_{1y}) \\ q_{2t} &= -q_{2xy} - q_2 \partial_x^{-1}(q_2 q_{1y} - q_1 q_{2y}) - 2 q_1 \partial_x^{-1}(q_2 q_{2y}) \end{aligned} \tag{8}$$

which are nonlinear evolution equations in (2 + 1) dimensions. Note that both (q_1, q_2) are fermionic.

We can also consider a larger Lie algebra $Sl(2/1)$ generated by the following set of commutation rules:

$$\begin{aligned} &[H_a, E_b]_- = K_{ab} E_b, \qquad [H_a, F_b]_- = -K_{ab} F_b \\ &[E_a, F_b]_\mp = \delta_{ab}, \qquad [H_a, G_b]_- = \Omega_{ab} G_b \\ &[E_1, E_2]_- = G, \qquad [E_1, G_2]_- = -F_2 \end{aligned} \tag{9}$$

$$[E_2, G_2]_+ = F_1, \qquad [F_1, F_2]_- = G_2, \qquad [F_1, G_1]_- = +E_2$$
$$[F_2, G_1]_+ = E_1, \qquad [G_1, G_2]_+ = H_1 + H_2$$

where

$$K_{ab} = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}, \qquad \Omega_{ab} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

The plus and minus signs denote the anti and usual commutators. Here we make the choice

$$k = \{H_1, H_2, E_2, F_2\} \tag{10}$$
$$m = \{G_1, G_2, E_1, F_1\}$$

Again we start from equation (4) and set

$$Q = q_1G_1 + q_2G_2 + q_3E_1 + q_4F_1$$

We also assume that

$$B_2^m = S_1G_1 + S_2G_2 + S_3E_1 + S_4F_1 \tag{11a}$$

whence $\partial_y Q = [H_1, B_2^m]$ implies that

$$S_1 = q_{1y}, \qquad S_2 = -q_{2y}, \qquad S_3 = \tfrac{1}{2}\, q_{3y}, \qquad S_4 = -\tfrac{1}{2}\, q_{4y}$$

Furthermore, from equation (4) we get

$$B_2^K = M_1H_1 + M_2H_2 + M_3E_2 + M_4F_2$$

with

$$M_1 = \partial_x^{-1}\{(q_2q_{1y} - q_1q_{2y}) - \tfrac{1}{2}\,(q_3q_4)_y\}$$
$$M_2 = \partial_x^{-1}(q_2q_{1y} - q_1q_{2y})$$
$$M_3 = \partial_x^{-1}(\tfrac{1}{2}\, q_1q_{4y} + q_4q_{1y})$$
$$M_4 = \partial_x^{-1}\{\tfrac{1}{2}\, q_2q_{3y} + q_3q_{2y}\}$$

So finally from equation (4c) the evolution equations turn out to be

$$q_{1t} = q_{1xy} + q_1M_1 - q_3M_3 - q_1M_2$$
$$q_{2t} = -q_{2xy} - q_2M_1 + q_2M_2 \quad q_4M_4 \tag{11b}$$
$$q_{3t} = \tfrac{1}{2}\, q_{3xy} - q_1M_4 - q_3M_2 + 2q_3M_1$$
$$q_{4y} = -\tfrac{1}{2}\, q_{4xy} - q_2M_3 + q_4M_2 - 2q_4M_1$$

In contrast to the previous situation, here we have both bosonic (q_1, q_2) and fermionic field variables.

It is interesting to note that another type of nonlinear equation can be generated by relaxing the conditions on the subspaces k and m. Let us suppose that

$$[m, m] \subset k \text{ or } m$$

instead of the previously assumed condition. Then for the algebra we have

$$\begin{aligned} g &= k \oplus m \\ k &= H \\ m &= \{P, R, E, F\} \end{aligned} \tag{12}$$

so that for $n = 2$ we go back to the basic set (3) and deduce

$$\partial_y Q = [A, B_2^m] \tag{13a}$$

$$\partial_x B_2^k = [Q, B_2^m]_k \tag{13b}$$

$$\partial_t Q = \partial_x B_2^m - [Q, B_2^k] - [Q, B_2^m]_m \tag{13c}$$

To have the explicit structure we set

$$Q = q_1 P + q_2 R + q_3 E + q_4 F \tag{14}$$

In the above equation $[A, B]_k$ denotes the projection of the commutator in the k subspace. Proceeding as before, we get

$$\begin{aligned} B_2^m &= q_{1y} P - q_{2y} R + \tfrac{1}{2}(q_{3y} E - q_{4y} F) \\ B_2^k &= \partial_x^{-1}\{-q_1 q_{2y} + q_2 q_{1y} - \tfrac{1}{2}(q_3 q_4)\} H \end{aligned} \tag{15}$$

So equation (13c) finally leads to

$$\begin{aligned} \partial_t q_1 &= q_{1xy} + \tfrac{1}{2} q_2 q_{3y} + q_3 q_{2y} + q_1\, \partial_x^{-1} \tilde{Q} \\ \partial_t q_2 &= -q_{2xy} - \tfrac{1}{2} q_1 q_{4y} - q_4 q_{1y} - q_2\, \partial_x^{-1} \tilde{Q} \\ \partial_t q_3 &= \tfrac{1}{2} q_{3xy} + 2 q_1 q_{1y} + 2 q_3\, \partial_x^{-1} \tilde{Q} \\ \partial_t q_4 &= -\tfrac{1}{2} q_{4xy} + 2 q_2 q_{2y} - 2 q_4\, \partial_x^{-1} \tilde{Q} \end{aligned}$$

where

$$\tilde{Q} = -q_1 q_{2y} + q_2 q_{1y} - \tfrac{1}{2}(q_3 q_4)_y$$

In the case of the algebra $Sl(2/1)$

$$k = \{H_1, H_2\} \tag{16a}$$

$$m = \{G_1, G_2, E_1, F_1, E_2, F_2\}$$

In this case

$$Q = q_1 G_1 + q_2 G_2 + q_3 E_1 + q_4 F_1 + q_5 E_2 + q_6 F_2 \tag{16b}$$

Without giving the details of the computation, we can just quote the final result. The nonlinear equations turn out to be

$$\begin{aligned}
q_{1t} &= q_{1xy} + q_3 q_{5y} + \tfrac{1}{2} q_5 q_{3y} + q_1(M - M') \\
q_{2t} &= -q_{2xy} + q_4 q_{6y} + \tfrac{1}{2} q_6 q_{4y} - q_2(M - M') \\
q_{3t} &= \tfrac{1}{2} q_{3xy} - q_1 q_{6y} - q_6 q_{1y} + q_3(2M - M') \\
q_{4t} &= -\tfrac{1}{2} q_{4xy} + q_2 q_{5y} + q_5 q_{2y} - q_4(2M - M') \\
q_{5t} &= -q_{5xy} - q_4 q_{1y} - \tfrac{1}{2} q_1 q_{4y} - q_5 M \\
q_{6t} &= q_{6xy} - q_3 q_{2y} - \tfrac{1}{2} q_2 q_{3y} + q_6 M
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
M &= \partial_x^{-1}(-q_1 q_{2y} + q_2 q_{1y} - \tfrac{1}{2} q_3 q_{4y} - \tfrac{1}{2} q_4 q_{3y}) \\
M' &= \partial_x^{-1}(-q_1 q_{2y} + q_2 q_{1y} + q_5 q_{6y} - q_6 q_{5y})
\end{aligned}$$

3. THE SECOND APPROACH

The form of the Lax pair used in the previous section was dictated by the fact that one needs to have the commutator of matrices in order to utilize the concept of homogeneous spaces. Thus we used a Lax operator depending both on the parameter ξ and the operator ∂y. But one can also use a two-dimensional Lax operator without any such spectral parameter.

This form of Lax operator was previously used by Zakharov, among others. One can also have supersymmetric equations in $(2 + 1)$ dimensions without using the concept of homogeneous space if this type of Lax operator is used. We write the two parts of the Lax operators as

$$T_1(Q) = \partial_x + H_1\partial_y + Q$$
$$T_2(Q) = i\partial_t + H_1\partial_y^2 + A\,\partial_y + B \tag{18}$$

and impose the condition

$$(T_1T_2 - T_2T_1)\Psi = 0 \tag{19}$$

We consider the case of the Lie algebra $SL(2/1)$. Evaluating (19) explicitly, we get

$$\begin{aligned}(T_1T_2 - T_2T_1)\Psi \\ &= (\partial_x B + H\partial_y B + QB - i\partial_t Q - H_1\,\partial_y^2 Q - A\,\partial_y Q - BQ)\Psi \\ &\quad +(\partial_x A + H_1\,\partial_y A + H_1 B + QA - 2H_1\,\partial_y Q - AQ - BH_1) \\ &\quad \times\partial_y\Psi + (H_1A + QH_1 - H_1Q - AH_1)\,\partial_y^2\Psi\end{aligned} \tag{20}$$

Equating to zero the coefficients of ψ, $\partial_y\psi$, $\partial_y^2\Psi$, we get

$$\partial_x B + H_1\partial_y B + [Q, B] - i\partial_t Q - H_1\,\partial_y^2 Q - A\partial_y Q = 0 \tag{21a}$$
$$\partial_x A + H_1\partial_y A + [H_1, B] + [Q, A] - 2H_1\partial_y Q = 0 \tag{21b}$$
$$[H, A] + [Q, H_1] = 0 \tag{21c}$$

Equation (21c) implies $Q = A$, whence from (21b) we get

$$\partial_x Q + H_1\partial_y Q = [H_1, B] \tag{22}$$

which determines the matrix B. We get

$$b_2 = \tfrac{1}{2}\,\partial_- q_3, \qquad b_3 = \partial_- q_1, \qquad b_4 = \tfrac{1}{2}\,\partial_+ q_4$$
$$b_6 = \partial_+ q_5, \qquad b_7 = q_{2x}, \qquad b_8 = -q_{6x}$$

where

$$\partial_- = \partial_y - \partial_x, \qquad \partial_+ = \partial_y + \partial_x \tag{23}$$

On the other hand, the diagonal elements of B are

$$\begin{aligned} b_1 &= +\partial_+^{-1}\,\partial_-(q_1q_2 + \tfrac{1}{2}\,q_3q_4) \\ b_5 &= -\partial_-^{-1}\,\partial_+(q_5q_6 + \tfrac{1}{2}\,q_3q_4) \\ b_9 &= q_2q_1 - q_6q_5 \end{aligned} \tag{24}$$

Once the matrices A and B are determined, one can write down the nonlinear systems from equation (21a):

$$
\begin{aligned}
iq_{3t} &= \partial_+ b_2 - \partial_y^2 q_3 - q_1 q_{6y} + q_3 b_5 + q_1 b_8 - b_1 q_3 - b_3 q_6 \\
iq_{1t} &= \partial_+ b_3 - \partial_y^2 q_1 - q_3 q_{5y} + q_3 b_6 + q_1 b_9 - b_1 q_1 - b_2 q_5 \\
iq_{4t} &= -\partial_- b_4 + \partial_y^2 q_4 - q_5 q_{2y} + q_4 b_1 + q_5 b_7 - b_5 q_4 - b_6 q_2
\end{aligned} \tag{25}
$$

with similar equations for (q_5, q_6, q_2). It is quite obvious that similar computations can also be performed for the case of any other super-Lie algebra.

4. BACKLUND TRANSFORMATION

The solutions of the nonlinear integrable systems derived above can be studied through the use of Backlund transformation. There are several ways to arrive at such results. One can use the discrete symmetry of the Lax operator, the method of Riccati equations, that of Gauss decomposition, and lastly that of Daboux transformation. Here we have adopted the last approach for the Lax operator (2). Suppose we have two sets of solutions of the same nonlinear system given as Q and Q', which occur as potentials in the corresponding linear problems,

$$
\begin{aligned}
\psi'_x &= (\lambda A + Q')\psi \\
\psi_x &= (\lambda A + Q)\psi
\end{aligned} \tag{26}
$$

In the Darboux approach one assumes that $\psi' = D\psi$ with $D = \lambda D_0 + D_1$. Equation (26) along with $\psi' = D\psi$ immediately leads to

$$D_x = (\lambda A + Q')D - D(\lambda A + Q) \tag{27}$$

Putting in this equation the form $D_0\lambda + D_1$ of D, we get [with the form of Q given in (16b)]

$$
\begin{aligned}
D_1 &= A \\
D_{0x} &= Q'D_0 - D_0 Q \\
[A, D_0] + Q'A - AQ &= D_{1x}
\end{aligned} \tag{28}
$$

The last and first equation of (28) at once lead to [for the set of equations (11), $A = H_1$]

$$
D_0^{\text{off}} = \begin{pmatrix} 0 & \frac{1}{2}(q_3 + q'_3) & q_1 \\ \frac{1}{2}(q_4 + q'_4) & 0 & q_5 \\ q'_2 & q'_6 & 0 \end{pmatrix} \tag{29}
$$

The elements D_0^{diag} are given by the diagonal part of the second equation (28), which is

$$D_0^{\text{diag}} = \begin{pmatrix} \partial_x^{-1} f(12, 34) & 0 & 0 \\ 0 & \partial_x^{-1} f(56, 34) & 0 \\ 0 & 0 & d \end{pmatrix} \tag{30}$$

where d is constant and

$$f(ij, kl) = q_i' q_j' - q_i q_j + \tfrac{1}{2}(q_k' q_l' - q_k q_l)$$

Next the off-diagonal part of the same equation yields

$$Q' \,(D_0^{\text{diag}} + D_0^{\text{off}}) - (D_0^{\text{diag}} + D_0^{\text{off}})Q = (D_0^{\text{off}})_x \tag{31}$$

which gives the explicit connection between the old and new variables Q, Q' when D_0^{diag} and D_0^{off} are substituted from equations (29) and (30). Equation (31) gives a set of six equations for the Backlund transformation in the $Sl(2/1)$ case discussed in equation (17). Incidentally it may be mentioned that such an approach does not hold in the case of the Lax operators given in (18).

For such types of Lax operators it is customary to write

$$\begin{aligned} T_1(Q) &= \partial_x + H_1\partial_y + Q \\ T_1(Q') &= \partial_x + H_1\partial_y + Q' \end{aligned} \tag{32}$$

where Q, Q' are given as [see (16b)]

$$\begin{aligned} T_2(Q) &= i\partial_t + H_1\partial_y^2 + A\partial_y + B \\ T_2(Q') &= i\partial_t + H_1\partial_y^2 + A'\partial_y + B' \end{aligned} \tag{33}$$

along with the condition

$$[T_1\ T_2] = 0$$

which implies

$$[T_1', T_2'] = 0$$

If ψ and ψ' denote the common eigenfunctions of the initial and final Lax pair, then the gauge transformation

$$\psi' = D(Q', Q)\psi \tag{34}$$

should exist such that

$$\begin{aligned} T_1(Q')D(Q', Q) - D(Q', Q)T_1(Q) &= 0 \\ T_2(Q')D(Q', Q) - D(Q', Q)T_2(Q) &= 0 \end{aligned} \tag{35}$$

To proceed further, set $D(Q', Q) = \alpha\partial_y + D_0(Q', Q)$ in the two equations of (35) and equate the coefficients of ψ and $\partial_y\psi$ to zero. This yields

$$\partial_x D_0 + H_1\, \partial_y D_0 - \alpha\partial_y Q + Q' D_0 - D_0 Q = 0 \tag{36}$$

$$[H_1, D_0] + Q'\alpha - \alpha Q = 0 \tag{37}$$

when the first condition of (35) is used. Here we have assumed

$$\alpha = \begin{pmatrix} \alpha_1 & & \\ & \alpha_2 & \\ & & \alpha_3 \end{pmatrix} = \text{diagonal constant matrix}$$

whence one gets

$$D^{\text{off}} = \begin{pmatrix} 0 & \frac{1}{2}(\alpha_1 q_3 - \alpha_2 q_3') & (\alpha_1 q_1 - \alpha_3 q_1' \\ \frac{1}{2}(\alpha_1 q_4' - \alpha_2 q_4) & 0 & \alpha_3 q_5' - \alpha_2 q_5 \\ \alpha_1 q_2' - \alpha_3 q_2 & \alpha_3 q_6 - \alpha_2 q_6' & 0 \end{pmatrix}$$

On the other hand, the diagonal part of (36) is given by

$$D^{\text{diag}} = \begin{pmatrix} \alpha_1 \partial_t^{-1} f(12, 34) & 0 & 0 \\ 0 & -\alpha_2\, \partial_-^{-1} f(56, 34) & 0 \\ 0 & 0 & \alpha\partial_x^{-1} f(12, 56) \end{pmatrix}$$

Finally, using this explicit expression for D_0 in the equation obtained from the off-diagonal parts of (36), one gets the explicit Backlund transformation.

5. CONCLUSION

In the above analysis we have shown how supersymmetric generalization of nonlinear equations in (2 + 1) dimensions can be constructed and analyzed through the explicit realization of the Backlund transformation. It is interesting to observe that the hierarchy of equations is different depending on the method adopted. Also, the methodology for the derivation of the Backlund transformation needs be different for the two different situations.

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